

Conformal motions in plane symmetric static spacetimes

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ABSTRACT: In this paper, conformal motions are studied in plane symmetric static spacetimes. The general solution of conformal Killing equations and the general form of the conformal Killing vector for these spacetimes are presented. All possibilities for the existence of conformal motions in these spacetimes are exhausted.

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1. Introduction and Motivation

In addition to isometries there are other types of motions also which are very useful as far as the four dimensional Lorentzian metrics, their properties and their applications to mathematical physics are concerned. Conformal motions or conformal Killing vectors (CKVs) are motions along which the metric tensor of a spacetime remains invariant upto a scale factor. A conformal vector field can be defined[1, 2] as a global smooth vector field ξ on a manifold, M , such that for the metric g_{ab} in any coordinate system on M

$$\xi_{a;b} = \phi g_{ab} + F_{ab}, \quad (1.1)$$

where $\phi : M \longrightarrow R$ is the smooth conformal function of ξ , $F_{ab} (= -F_{ba})$ is the conformal bivector of ξ . This is equivalent to

$$\mathcal{L}_\xi g_{ab} = 2g_{ab}\phi(t, x, y, z), \quad (1.2)$$

where \mathcal{L}_ξ represents the Lie derivative with respect to ξ .

In explicit form we can write the above equation as,

$$g_{ab,c}\xi^c + g_{cb}\xi^c{}_{,a} + g_{ac}\xi^c{}_{,b} = 2g_{ab}\phi(t, x, y, z). \quad (1.3)$$

The “,” represents the partial derivative with respect to coordinates x^a ($x^0 = t, x^1 = x, x^2 = y, x^3 = z$). Here, if ϕ is constant, ξ are called homothetic motions or homothetic vector (HV) fields, and if it is zero we get Killing vector (KV) fields. The KVs, HVs and CKVs form finite dimensional Lie algebras. For a basis $\{\mathbf{X}_i, i = 1, \dots, n\}$ of a Lie algebra, we can always write the Lie bracket

$$[\mathbf{X}_k, \mathbf{X}_l] = C_{kl}^j \mathbf{X}_j \quad C_{kl}^j = -C_{lk}^j, \quad (1.4)$$

which is bilinear, antisymmetric and satisfies the Jacobi identity. Here C_{kl}^j are the structure constants which completely characterize the Lie algebra.

Here we state some of the well-known results[1, 2] regarding dimensionality of the Lie algebras for motions.

Theorem 1:

The Lie algebras of KVs and HVs are finite dimensional. For an n -dimensional manifold M admitting a metric of any signature, the dimension of the algebra of KVs is $\leq \frac{n(n+1)}{2}$ and that of HVs is $\leq \frac{n(n+1)}{2} + 1$.

Theorem 2:

The set of conformal vector fields on a spacetime is finite-dimensional and its dimension is ≤ 15 . If this maximum number is attained, the spacetime is conformally flat. If it is not conformally flat then the dimension of the set of CKVs is ≤ 7 .

Conformal motions are determined by the arbitrary constants appearing in the vector field $\xi = \xi^a \partial / \partial x^a$ when $\phi = \phi(t, x, y, z)$. The study of the symmetry groups of a spacetime is a useful tool in constructing spacetime solutions of EFEs and also classifying the known solutions according to their Lie algebras, or structures generated by these symmetries. They have physical significance as they generate motion along null geodesics for massless particles. CKVs have been studied for various classes of spacetimes including Minkowski[3], Friedmann-Robertson-Walker[4] and pp-waves[5]. The general solution and classification of conformal motions in static spherical spacetimes has also been carried out[6]. While much work has been done on isometries and homotheties, comparatively little is known about conformal symmetries. This is at least in part, due to the difficulty in solving the conformal Killing equations which contain conformal factor ϕ , which is in general a function of all coordinates[7].

In this paper we study the conformal motions or CKVs of plane symmetric static spacetimes. We first find the general solution of the equations of conformal motions in these spacetimes, together with the general form of the CKV and the conformal factor. Then we provide the complete classification of CKVs.

In Cartesian coordinates the general form of static plane symmetric spacetime is

$$ds^2 = e^{2\nu(x)} dt^2 - dx^2 - e^{2\mu(x)} (dy^2 + dz^2) \quad (1.5)$$

The general form of KVs, \mathbf{K} , for this metric is given by[8]

$$K^0 = e^{2(\mu-\nu)} \left\{ \frac{1}{2} \dot{A}_1(t, x) (z^2 + y^2) + \dot{A}_2(t, x) z + \dot{A}_3(t, x) y \right\} + K(t, x), \quad (1.6)$$

$$K^1 = -e^{2\mu} \left\{ \frac{1}{2} A'_1(t, x) (z^2 + y^2) + A'_2(t, x) z + A'_3(t, x) y \right\} + L(t, x), \quad (1.7)$$

$$K^2 = +\frac{c_3}{2} (z^2 - y^2) + yz c_2 + c_1 z + A_1(t, x) y + A_3(t, x), \quad (1.8)$$

$$K^3 = +\frac{c_2}{2} (z^2 - y^2) - yz c_3 - c_1 y + A_1(t, x) z + A_2(t, x), \quad (1.9)$$

where $A_i(t, x)$, $K(t, x)$ and $L(t, x)$ satisfy some differential constraints.

This classification reproduces the well-known static plane symmetric solutions, giving anti-de Sitter metric in the form

$$ds^2 = e^{\frac{2x}{x_0}} (dt^2 - dz^2 - dy^2) - dx^2. \quad (1.10)$$

It also reproduces the 5-dimensional isometries

$$\begin{aligned} K^0 &= x_0 \nu' c_5 + c_1, \\ K^1 &= -c_5 x_0, \\ K^2 &= c_5 y + c_4 z + c_2, \\ K^3 &= c_5 z - c_4 y + c_3, \end{aligned}$$

admitted by the class of metrics

$$ds^2 = e^\nu dt^2 - dx^2 - e^{\frac{x}{x_0}} (dz^2 + dy^2), \nu'' \neq 0. \quad (1.11)$$

There were three new metrics with 6 isometries. These were planar analogues of the Bertotti-Robinson metrics and two other similar metrics[9]. They have $\mu(x) = 0$ and

$$\begin{aligned} \nu(x) &= \ln \cosh^2 \alpha x, \\ \nu(x) &= e^{2\alpha x}, \\ \nu(x) &= \ln \cos^2 \alpha x, \end{aligned}$$

and the corresponding KVs are

$$\begin{aligned} K &= [c_0 - \tanh \alpha x (c_4 \sin \alpha t - c_5 \cos \alpha t)] \partial/\partial t + (c_4 \sin \alpha t + c_5 \cos \alpha t) \partial/\partial x \\ &\quad + (c_1 + c_3 z) \partial/\partial y + (c_2 - c_3 y) \partial/\partial z, \\ K &= [c_0 - \alpha (c_4 t^2 + c_5 t)] \partial/\partial t + (2c_4 t + c_5) \partial/\partial x + (c_1 + c_3 z) \partial/\partial y \\ &\quad + (c_2 - c_3 y) \partial/\partial z, \\ K &= [c_0 - (c_4 \sin \alpha t + c_5 \cos \alpha t) \tan \alpha x] \partial/\partial t + (c_4 \cos \alpha t + c_5 \sin \alpha t) \partial/\partial x \\ &\quad + (c_1 + c_3 z) \partial/\partial y + (c_2 - c_3 y) \partial/\partial z, \end{aligned}$$

where c_i are arbitrary constants.

2. General Form of Conformal Motions

The conformal Killing Eqs. (1.3) for the metric given by (1.5) represent a system of ten coupled non homogeneous first order partial differential equations. As the metric is diagonal, we note that from Eqs. (1.3) for $a = b$ the only non-zero contributions

from the second and the third term on the left hand side will come when $c = a = b$. When $a \neq b$ these terms will contribute only when $c = b$ and $c = a$, respectively. Thus by dropping the summation convention these can be written as

$$g'_{aa}\xi^1 + 2g_{aa}\xi_{,a}^a = 2g_{aa}\phi(t, x, y, z), \quad (a = 0, 1, 2, 3) \quad (2.1)$$

$$g_{bb}\xi_{,a}^b + g_{aa}\xi_{,b}^a = 0, \quad (a, b = 0, 1, 2, 3; a \neq b). \quad (2.2)$$

Here Eqs. (2.1) give four equations and Eqs. (2.2) are six equations. These are to be solved[10] to give six unknown functions, $\nu(x)$, $\mu(x)$ and $\xi^\alpha = \xi^\alpha(x^a)$ and the conformal factor $\phi(x^i)$. We first write Eqs. (2.2) for $a = 2, b = 0$ and $a = 3, b = 0$ and differentiate them with respect to z and y respectively to obtain

$$e^{2\nu}\xi_{,23}^0 - e^{2\mu}\xi_{,03}^2 = 0, \quad (2.3)$$

$$e^{2\nu}\xi_{,23}^0 - e^{2\mu}\xi_{,02}^3 = 0. \quad (2.4)$$

Now we write Eq. (2.2) for $a = 2, b = 3$ and differentiate with respect to t to get

$$\xi_{,03}^2 = -\xi_{,02}^3. \quad (2.5)$$

Using Eq. (2.5) in Eq. (2.4), we get

$$e^{2\nu}\xi_{,23}^0 + e^{2\mu}\xi_{,03}^2 = 0. \quad (2.6)$$

Adding Eqs. (2.3) and (2.6) we get

$$\xi_{,23}^0 = 0. \quad (2.7)$$

Similarly, from Eqs. (2.2) for $a = 1, b = 2$ and $a = 1, b = 3$ we get

$$\xi_{,23}^1 = 0. \quad (2.8)$$

Comparing Eqs. (2.1) for $a = 1$ and $a = 2$ gives

$$\xi_{,3}^2 = B_1(t, x, z)y + B_2(t, x, z), \quad (2.9)$$

where B_1 and B_2 are functions of integration. Using this in Eqs. (2.2) for $a = 2, b = 3$ and integrating with respect to y yields

$$\xi^3 = -\frac{1}{2}B_1(t, x, z)y^2 - B_2(t, x, z)y + B_3(t, x, z). \quad (2.10)$$

Substituting from Eqs. (2.9) and (2.10) in Eqs. (2.1) for $a = 2$ and $a = 3$, respectively, comparing the coefficients of y^2 , y and the terms independent of y , and integrating with respect to z yields

$$B_1(t, x, z) = F_1(t, x)z + F_2(t, x), \quad (2.11)$$

$$B_2(t, x, z) = F_3(t, x)z + F_4(t, x), \quad (2.12)$$

$$B_3(t, x, z) = F_1(t, x)\frac{z^3}{6} + F_2(t, x)\frac{z^2}{2} + A_1(t, x) + A_2(t, x), \quad (2.13)$$

where $F_i(t, x)$ and $A_i(t, x)$ are functions of integration. Using these values of B_1, B_2 and B_3 we obtain from Eqs. (2.9)- (2.10)

$$\xi^2 = \left\{ \frac{1}{2}F_1(t, x)z^2 + F_2(t, x)z \right\} y + \frac{1}{2}F_3(t, x)z^2 + F_4(t, x) + B_4(t, x, y), \quad (2.14)$$

and

$$\begin{aligned} \xi^3 = & -\frac{1}{2}\{F_1(t, x)z + F_2(t, x)\}y^2 - \{F_3(t, x)z + F_4(t, x)\}y \\ & + F_1(t, x)\frac{z^3}{6} + F_2(t, x)\frac{z^2}{2} + A_1(t, x) + A_2(t, x). \end{aligned} \quad (2.15)$$

where B_4 is a function of integration. Substituting these values of ξ^2 and ξ^3 in Eqs. (2.1) for $a = 2$ and $a = 3$ and comparing yields

$$B_4(t, x, y) = -\frac{1}{6}F_1(t, x)y^3 - \frac{1}{2}F_3(t, x)y^2 + A_1(t, x)y + A_3(t, x), \quad (2.16)$$

where $A_3(t, x)$ is a function of integration. Substituting ξ^3 in Eqs. (2.2) for $a = 0, b = 3$ and $a = 1, b = 3$ respectively and integrating the resulting equations yields

$$\begin{aligned} \xi^0 = & e^{2(\mu-\nu)}[-\{\frac{1}{4}\dot{F}_1(t, x)z^2 + \frac{1}{2}\dot{F}_2(t, x)z\}y^2 - \{\frac{1}{2}\dot{F}_3(t, x)z^2 + \dot{F}_4(t, x)z\}y \\ & + \frac{1}{24}\dot{F}_1(t, x)z^4 + \frac{1}{6}\dot{F}_2(t, x)z^2 - \dot{A}_2(t, x) + \frac{1}{2}\dot{A}_1(t, x)y^2 + \dot{A}_3(t, x)y] + A_0(t, x) \end{aligned} \quad (2.17)$$

$$\begin{aligned} \xi^1 = & -e^{2\mu}[-\{\frac{1}{4}F'_1(t, x)z^2 + \frac{1}{2}F'_2(t, x)z\}y^2 - \{\frac{1}{2}F'_3(t, x)z^2 + F'_4(t, x)z\}y \\ & + \frac{1}{24}F'_1(t, x)z^4 + \frac{1}{6}F'_2(t, x)z^2 - A'_2(t, x) + \frac{1}{2}A'_1(t, x)y^2 + A'_3(t, x)y] + A_4(t, x) \end{aligned} \quad (2.18)$$

where $A_0(t, x)$ and $A_4(t, x)$ are functions of integration.

Substituting these values of ξ^i in Eqs. (2.1) and (2.2) for checking consistency yields

$$\dot{F}_i(t, x) = 0 = F'_i(t, x), \quad (i = 1, 2, 3, 4). \quad (2.19)$$

Thus

$$F_1(t, x) = c_0, F_2(t, x) = c_2, F_3(t, x) = c_3, F_4(t, x) = c_4, \quad (2.20)$$

where c_i , are arbitrary constants. Hence we obtain the general form of the CKV as

$$\begin{aligned} \xi = & \left[e^{2(\mu-\nu)} \left\{ \frac{1}{2} \dot{A}_1(t, x) (z^2 + y^2) + \dot{A}_2(t, x) z + \dot{A}_3(t, x) y \right\} + A_0(t, x) \right] \partial/\partial t \\ & - \left[e^{2\mu} \left\{ \frac{1}{2} A'_1(t, x) (z^2 + y^2) + A'_2(t, x) z + A'_3(t, x) y \right\} - A_4(t, x) \right] \partial/\partial x \\ & + \left[\frac{c_3}{2} (z^2 - y^2) + c_2 y z + c_4 z + A_1(t, x) y + A_3(t, x) \right] \partial/\partial y \\ & + \left[\frac{c_2}{2} (z^2 - y^2) - c_3 y z - c_4 y + A_1(t, x) z + A_2(t, x) \right] \partial/\partial z. \end{aligned}$$

In order to obtain the explicit form of ξ^a from the above we need to know the arbitrary functions $A_i(t, x)$, $i = 0, \dots, 4$. For this we substitute these equations in the CKV equations (2.1) and (2.2) and obtain the following differential constraints[10].

$$A''_1(t, x) + \mu' A'_1(t, x) = 0, \quad (2.21)$$

$$A''_2(t, x) + \mu' A'_2(t, x) = -c_2 e^{-2\mu}, \quad (2.22)$$

$$A''_3(t, x) + \mu' A'_3(t, x) = c_3 e^{-2\mu}, \quad (2.23)$$

$$A'_k(t, x) [2\mu' - \nu'] + e^{-2\nu} \ddot{A}_k(t, x) + A''_k(t, x) = 0, \quad k = 1, 2, 3 \quad (2.24)$$

$$\dot{A}'_k(t, x) + [\mu' - \nu'] \dot{A}_k(t, x) = 0, \quad k = 1, 2, 3 \quad (2.25)$$

$$\mu' A_4(t, x) + A_1(t, x) - A'_4(t, x) = 0, \quad (2.26)$$

$$e^{2\nu} A'_0(t, x) - \dot{A}_4(t, x) = 0, \quad (2.27)$$

$$\nu' A_4(t, x) + \dot{A}_0(t, x) - A'_4(t, x) = 0. \quad (2.28)$$

We also find that the general form of the conformal factor is

$$\begin{aligned} \phi = & -e^{2\mu} \left[\frac{1}{2} A''_1(t, x) (z^2 + y^2) + A''_2(t, x) z + A''_3(t, x) y \right] \\ & - 2\mu' e^{2\mu} \left[\frac{1}{2} A'_1(t, x) (z^2 + y^2) + A'_2(t, x) z + A'_3(t, x) y \right] + A'_4(t, x). \end{aligned} \quad (2.29)$$

3. Classification of Conformal Motions

The problem of finding CKVs in plane symmetric static spacetimes is now reduced to solving the twelve coupled non linear non homogeneous second order partial differential Eqs. (2.21)-(2.28) to give seven unknown functions ν , μ and $A_i(t, x)$, $i = 0, \dots, 4$. The arbitrary constants in ξ will determine the number of generators of the Lie algebra. We divide our classification scheme into different cases depending upon whether one, both or none of the metric coefficients are constants[10].

Let us first consider the simplest case when both μ and ν are constant. We note that the Weyl tensor becomes zero and thus this is a class of conformally flat spacetimes. The final form of the CKVs is

$$\begin{aligned}\xi^0 &= \left[\frac{1}{2}c_5 (z^2 + y^2) + (c_2t + c_7)z + (-c_3t + c_{11})y \right] + c_1xt + c_1 \left(\frac{x^2}{2} + \frac{t^2}{2} \right) + \\ &\quad c_6t + c_{13}x + c_{15}, \\ \xi^1 &= - \left[\frac{1}{2}c_1 (z^2 + y^2) + (-c_2x + c_9)z + (c_3x + c_{10})y \right] + c_5tx + c_1 \left(\frac{x^2}{2} + \frac{t^2}{2} \right) + \\ &\quad c_6x + c_{13}t + c_{14}, \\ \xi^2 &= \frac{c_3}{2} (z^2 - y^2) + c_2yz + c_4z + (c_1x + c_5t + c_6)y + c_3\frac{x^2}{2} + c_{10}x - \\ &\quad c_3\frac{t^2}{2} + c_{11}t + c_{12}, \\ \xi^3 &= \frac{c_2}{2} (z^2 - y^2) - c_3yz - c_4y + (c_1x + c_5t + c_6)z + c_2 \left(\frac{t^2}{2} - \frac{x^2}{2} \right) + \\ &\quad c_7x + c_7t + c_8,\end{aligned}$$

which is the well-known[3] 15-dimensional Lie algebra of Minkowski spacetime. The coformal factor from Eq. (2.29) takes the following form in this case

$$\phi(t, x, y, z) = c_5t + c_1x - c_3y + c_2z + c_6. \quad (3.1)$$

Next we take $\mu' \neq 0$, $\nu' = 0$, and for simplicity we take $\nu = 0$. It is worth noting here that the metric in this case is conformally related to the one where both μ' and ν' are non-zero as we can redefine the coordinate x to express the metric in former way. The conformal algebra in the two cases are identical. However, when $\mu' = \nu' \neq 0$, the spacetime is conformally related to Minkowski spacetime and therefore admits a 15-dimensional conformal algebra.

Substituting $\nu = 0$ in Eqs. (2.21)-(2.28), we have

$$A_1''(t, x) + \mu' A_1'(t, x) = 0, \quad (3.2)$$

$$A_2''(t, x) + \mu' A_2'(t, x) = -c_2 e^{-2\mu}, \quad (3.3)$$

$$A_3''(t, x) + \mu' A_3'(t, x) = c_3 e^{-2\mu}, \quad (3.4)$$

$$\dot{A}_k'(t, x) + \mu' \dot{A}_k(t, x) = 0, \quad k = 1, 2, 3 \quad (3.5)$$

$$2\mu' A_k'(t, x) + \ddot{A}_k(t, x) + A_k''(t, x) = 0, \quad k = 1, 2, 3 \quad (3.6)$$

$$\mu' A_4(t, x) + A_1(t, x) - A_4'(t, x) = 0, \quad (3.7)$$

$$A_0'(t, x) - \dot{A}_4(t, x) = 0, \quad (3.8)$$

$$\dot{A}_0(t, x) - A_4'(t, x) = 0. \quad (3.9)$$

For $k = 1$ Eq. (3.5) can be written as

$$\left[e^\mu \dot{A}_1(t, x) \right]' = 0. \quad (3.10)$$

Integrating Eq. (3.10) with respect to x , and then with respect to t yields

$$A_1(t, x) = f_1(t) e^{-\mu} + g_1(x), \quad (3.11)$$

where $f_1(t)$ and $g_1(x)$ are functions of integration.

Similarly for $k = 2$, Eq. (3.5) can be written as

$$A_2(t, x) = f_2(t) e^{-\mu} + g_2(x). \quad (3.12)$$

Differentiating Eq. (3.11) twice with respect to x gives

$$A_1''(t, x) = (\mu'^2 - \mu'') e^{-\mu} f_1(t) + g_1''(x). \quad (3.13)$$

Substituting the values $A_1'(t, x)$ and $A_1''(t, x)$ in Eq. (3.2) we get after some simplification

$$\mu'' f_1(t) = [e^\mu g_1'(x)]', \quad (3.14)$$

Differentiating Eq. (3.14) with respect to t we get

$$\mu'' \dot{f}_1(t) = 0. \quad (3.15)$$

This equation gives rise to two cases: Either $\mu'' = 0$ or not. In the first case we let $\mu = ax + b$, so that the general form of the metric is

$$ds^2 = dt^2 - dx^2 - e^{(ax+b)}(dy^2 + dz^2). \quad (3.16)$$

It is a conformally flat spacetime having 15-dimensional Lie algebra.

When $\dot{f}_1(t) = 0$ we put $f_1(t) = d_1$, a constant, so that Eq. (3.14) on integrating twice with respect to x gives

$$g_1(x) = d_1 \int e^{-\mu} \mu' dx + d_2 \int e^{-\mu} dx + d_3. \quad (3.17)$$

Inserting the values of $f_1(t)$ and $g_1(x)$ in Eq.(3.11), we obtain

$$A_1(t, x) = d_2 \int e^{-\mu} dx + d_3. \quad (3.18)$$

Using this value in Eq. (3.6) gives

$$A_1(t, x) = d_3. \quad (3.19)$$

Substituting the value of $A_2(t, x)$ from Eq. (3.12) in Eq. (3.3) and keeping in view that $\mu'' \neq 0$ we see that

$$A_2(t, x) = d_4. \quad (3.20)$$

Similarly,

$$A_3(t, x) = d_5. \quad (3.21)$$

Eliminating $A_0(t, x)$ between Eq. (3.8) and Eq.(3.9) and using Eq. (3.7) shows that

$$A_4(t, x) = A_1(t, x) = 0. \quad (3.22)$$

Finally, Eq. (3.8) and (3.9) yield

$$A_0(t, x) = c_1. \quad (3.23)$$

Thus the conformal factor becomes zero and we get (calling d_4 and d_5 as c_3 and c_2 respectively)

$$\begin{aligned} \xi^0 &= c_1, \\ \xi^1 &= 0, \\ \xi^2 &= c_4 z + c_2, \\ \xi^3 &= -c_4 y + c_3, \end{aligned}$$

which is a 4-dimensional Killing algebra representing the minimal isometry for plane symmetric static spacetimes.

Now, we consider the case when $\nu' \neq 0$ and $\mu' = 0$. Proceeding in the same fashion as before, we see that the constraint equations (Eqs. (2.21)-(2.28)) give

rise to two possibilities: Either $(\nu''e^{2\nu})'$ is zero or not. In the first case we write $\nu''e^{2\nu} = k_1$, a constant, and find that $\phi = 0$, giving the KVs as

$$\begin{aligned}\xi^0 &= \left(c_5 \cos \sqrt{k_1}t + c_6 \sin \sqrt{k_1}t\right) \int e^{-2\nu} dx + c_1, \\ \xi^1 &= \frac{1}{\sqrt{k_1}} \left(c_5 \sin \sqrt{k_1}t - c_6 \cos \sqrt{k_1}t\right), \\ \xi^2 &= c_4 z + c_2, \\ \xi^3 &= -c_4 y + c_3.\end{aligned}$$

The six dimensional Killing algebra is given by

$$\begin{aligned}[\mathbf{X}_1, \mathbf{X}_2] &= 0, & [\mathbf{X}_1, \mathbf{X}_3] &= 0, & [\mathbf{X}_1, \mathbf{X}_4] &= 0, \\ [\mathbf{X}_1, \mathbf{X}_5] &= -k_2 \mathbf{X}_6, & [\mathbf{X}_1, \mathbf{X}_6] &= k_2 \mathbf{X}_5, & [\mathbf{X}_2, \mathbf{X}_3] &= 0, \\ [\mathbf{X}_2, \mathbf{X}_3] &= 0, & [\mathbf{X}_2, \mathbf{X}_4] &= -\mathbf{X}_3, & [\mathbf{X}_2, \mathbf{X}_5] &= 0, \\ [\mathbf{X}_2, \mathbf{X}_6] &= 0, & [\mathbf{X}_3, \mathbf{X}_4] &= \mathbf{X}_2, & [\mathbf{X}_3, \mathbf{X}_5] &= 0, \\ [\mathbf{X}_3, \mathbf{X}_6] &= 0, & [\mathbf{X}_4, \mathbf{X}_5] &= 0, & [\mathbf{X}_4, \mathbf{X}_6] &= 0, \\ [\mathbf{X}_5, \mathbf{X}_6] &= k_3 \mathbf{X}_1.\end{aligned}$$

Here $k_2 = (x + \nu'/\nu'')$ and $k_3 = (\sqrt{k_1} \left(\int e^{-2\nu} dx\right)^2 + \frac{1}{\sqrt{k_1}} e^{-2\nu})$ are constants.

We note that when ν'' is zero (i.e. ν' is constant) we obtain 5 KVs given by

$$\begin{aligned}\xi^0 &= -\nu' c_5 t + c_1, \\ \xi^1 &= c_5, \\ \xi^2 &= c_4 z + c_2, \\ \xi^3 &= -c_4 y + c_3.\end{aligned}$$

The Lie algebra in this case is

$$\begin{aligned}[\mathbf{X}_1, \mathbf{X}_2] &= 0, & [\mathbf{X}_1, \mathbf{X}_3] &= 0, & [\mathbf{X}_1, \mathbf{X}_4] &= 0, \\ [\mathbf{X}_1, \mathbf{X}_5] &= -\nu' \mathbf{X}_1, & [\mathbf{X}_2, \mathbf{X}_3] &= 0, & [\mathbf{X}_2, \mathbf{X}_4] &= -\mathbf{X}_3, \\ [\mathbf{X}_2, \mathbf{X}_5] &= 0, & [\mathbf{X}_3, \mathbf{X}_4] &= \mathbf{X}_2, & [\mathbf{X}_3, \mathbf{X}_5] &= 0, \\ [\mathbf{X}_4, \mathbf{X}_5] &= 0.\end{aligned}$$

In the other case ν'' is not zero, and we either get minimal symmetry for the plane or six dimensional homotheties with $\phi = c_4$.

$$\begin{aligned}\xi^0 &= k c_5 t + c_1, \\ \xi^1 &= c_5 x, \\ \xi^2 &= c_4 z + c_5 y + c_2, \\ \xi^3 &= -c_4 y + c_5 z + c_3.\end{aligned}$$

The Lie algebra is given by

$$\begin{aligned} [\mathbf{X}_1, \mathbf{X}_2] &= 0, & [\mathbf{X}_1, \mathbf{X}_3] &= 0, & [\mathbf{X}_1, \mathbf{X}_4] &= 0, \\ [\mathbf{X}_1, \mathbf{X}_5] &= k\mathbf{X}_1, & [\mathbf{X}_2, \mathbf{X}_3] &= 0, & [\mathbf{X}_2, \mathbf{X}_4] &= -\mathbf{X}_3, \\ [\mathbf{X}_2, \mathbf{X}_5] &= \mathbf{X}_2, & [\mathbf{X}_3, \mathbf{X}_4] &= \mathbf{X}_2, & [\mathbf{X}_3, \mathbf{X}_5] &= \mathbf{X}_3, \\ [\mathbf{X}_4, \mathbf{X}_5] &= 0. \end{aligned}$$

Here $k = -\nu'x + \nu'k_1 + 1$ is a constant.

4. Conclusion

Conformal motions are the vectors along which the metric tensor of a spacetime remains invariant upto a factor, called the conformal factor. If this factor is constant then the symmetry is called an HV and if it is zero we get KVs. Therefore HVs and KVs are special cases of CKVs. Here we have classified plane symmetric static spacetimes according to conformal motion (or CKVs). We have solved conformal Killing equations, which are first order nonhomogeneous differential equations, to construct the general form of CKVs and the conformal factor ϕ along with a set of constraint equations. To simplify the classification scheme we divide it into cases depending upon whether the metric coefficients $\nu(x)$ and $\mu(x)$ are constant or not.

When both ν and μ are constant, we get flat spacetime admitting a maximal of 15 CKVs with the conformal factor given by

$$\phi(t, x, y, z) = c_5t + c_1x - c_3y + c_2z + c_6. \quad (4.1)$$

In case when $\nu' = 0$, but $\mu' \neq 0$, we get a 4-dimensional minimal Killing algebra for static plane symmetry or the metric becomes conformally flat. The metric in this case and the one when both ν' and μ' are nonzero, are conformally related and give the same conformal algebra. However, when $\nu' = \mu' \neq 0$, we get a flat spacetime with 15-dimensional algebra.

When we take $\nu' \neq 0$, and $\mu' = 0$, we obtain 5- or 6-dimensional Killing algebras or 6-dimensional homothety algebra.

Thus we conclude that plane symmetric static spacetimes do not admit non-trivial conformal motions apart from KVs or HVs, or the conformally flat cases.

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